Some Considerations in the Theories of Combinations, Probabilities, and Life Contingencies, by Peter Hardy, Esq., F.R.S. *and* F.I.A., *Actuary to the London Assurance Corporation.*

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My object in the following paper is to give, in a few leading problems, a general and somewhat elementary view of the Theory of Probabilities, exhibiting the immediate dependence of the doctrine of Life Contingencies on the laws of probability and combination, and showing how apparently complicated problems, involving two and three lives, can be deduced directly from their very simplest and most elementary forms of expression.

From the nature of the subject-matter proposed to be treated, it will be at once perceived, that this paper is intended rather for the instruction of the student in this branch of mathematical analysis, than for the use of the proficient; and I claim for it no higher distinction.

By the word "probability," it is intended to signify, not any absolute tendency which a given event has to happen or to fail, but merely an excess or an *equality* in the number of ways in which it is possible that such event can happen or can fail. It will be convenient to designate the *ways* in which an event can happen or can fail, by the word "*possibilities,*" consequently, if all the possibilities are in favour of the happening of the event, *it is impossible that it should fail; i. e.* it must happen, or *vice versâ.* If also the possibilities be equally divided, one half being in favour of the happening, and the other half being in favour of the failure of the event, then the probabilities will be equal also.

The word "chance" was formerly employed by mathematical writers to signify this excess or equality, but the use of the word in this sense often leads to misconceptions. We are accustomed to regard the word "chance" as opposed to the word "design," as a result wholly accidental and unforeseen contrasted with a result pre-organized or arranged; as thus, if in passing through a street we should observe a piece of money lying on a door-step, we should ascribe its being there to the result of accident or chance; but if, in passing on, we should observe a similar piece of money lying in a similar position, on every door-step in the street, we should then, with some show of reason, conjecture that this was

not the result of mere accident or *chance,* but of intention or *design.*

Probability is, in truth, a measure of "belief," or "judgment," founded on the possession of certain data, some of which are in favour of one sort of judgment or belief, and others in favour of an opposite judgment or belief.

The data on which our judgment is to be formed ought to *exhaust* every "possibility," or way in which the event *can* happen; and no one "possibility" ought to be included in or dependent on another.

In an elegant paper on this subject, published some time since in the *Philosophical Magazine,* Mr. Donkin proposed to employ the two words "*exhaustive*" and "*exclusive,*" to imply these absolutely necessary conditions. The following anecdote will serve to illustrate the importance of having careful reference to them. This question was proposed, in the course of conversation, to the Author and some other actuaries, when in Scotland last year:—Two observations have been contemporaneously taken as to the time of day, by two independent and equally credible and competent observers, one of whom makes the time 10 minutes *before* 12 o'clock, while the other makes it 10 minutes *after* 12 o'clock; why, under these circumstances, should mathematicians call the time 12 *o'clock,—and why should it not be any other time which could be named?* A moment's reflection on the terms in which the question was proposed will show that, in the mind of the inquirer, the data were not necessarily considered as *exhaustive;* that they really were so, would seem sufficiently obvious. If no other observations were taken, *there could be no other "possibilities"* than the two given; and as they were equally *exclusive,* that is to say, the one forbade the other, judgment or belief would incline equally to both, and would lie exactly and evenly between the two: *it was therefore more likely to be* 12 *o'clock exactly, than either of the two observed times.*

Another example, as applying this mode of reasoning to the probabilities of life, will perhaps put the question before the reader in a stronger point of view.

If out of 1000 persons of one common age, alive at the commencement of a stated year, it were known that 900 lived to the beginning of the next year, and that 100 died in the interim, no mathematician would hesitate to say, that the probability, at the outset of the year, of any *one* out of the original 1000 persons existing through that year, was $\frac{900}{1000}$ or $\frac{9}{10}$, and the probability of

the same individual dying in the course of that year was $\frac{100}{1000}$ or $\frac{1}{10}$. Here the data are both "*exclusive*" and "*exhaustive:*" exclusive, because no one of those who survived could possibly be one of those who died, or *vice versâ;* and exhaustive, because all the original 1000 possibilities were accounted for and included in the number who survived (900) and in the number who died (100); hence a correct judgment is speedily formed by comparing the quantity of one possibility with the quantity of the other, and we assign without difficulty the measures $\frac{9}{10}$ and $\frac{1}{10}$ as the respective probabilities of a single life surviving over, or failing in, the year. But should it have happened that, out of 1000 individuals known to be alive at the commencement of the year, 800 only were *ascertained* to have survived the year, and 50 only were *known* to have died within that time, we could not form the same, nor indeed any accurate judgment on the subject; inasmuch as the possibilities in favour of living, and the possibilities in favour of dying, do not together exhaust the total possibilities originally given.

Assuming, however, that all proper conditions obtain, the measure of our belief in the happening or failure of an event will be expressed by a fraction, the numerator of which is the number of possibilities in favour of either its happening or its failure, and the denominator of which is the total number of possibilities originally given; for example, if there be altogether a possible ways in which an event may happen or fail, and out of that number a_1 is in favour of its happening, then the fraction which measures the amount of our belief that the event will happen, or, in other words, the "probability" of its happening, will be $\frac{a_1}{a}$, and the measure of *our disbelief that it will happen*, or, in other words, the probability that it will fail, will be $\frac{a-a_1}{a} = 1 - \frac{a_1}{a}$; which form of expression brings us to another consideration.

It is obvious that, whatever may be the total number of possibilities, if one part of them favour the happening of the event, and if all the rest favour its failure,—that is, if the possibilities be exhaustive,—the sum of the two fractions measuring the two sorts of probability will always be equal to unity, as thus: $\frac{a_1}{a} + 1 - \frac{a_1}{a} = 1$. We learn, therefore, to consider unity as the representa-

tive of certainty, and, *è contra,* it is equally obvious that 0 may be put equal to impossibility.

When two, three, or more independent events are under consideration together, the product of the two, three, or more fractions separately indicating their respective probabilities will represent the new probability which arises of their both or all happening together, or in succession.

If we toss a piece of money, the probability that we shall throw a head will be manifestly one out of two, or $\frac{1}{2}$; and the further probability that we throw a head twice in succession will be

$$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4},$$

the measure of the probability required. This may be made sufficiently obvious : as thus, we may throw—

First Trial.	Second Trial.
H	H
H	T
T	H
T	T

Now, out of these four results, which are exhaustive of the possibilities, it is manifest that only one possibility, viz., the first, favours the repetition of head in two throws.

Let the question now be to throw head three times successively, and we shall have—

First Trial.	Second Trial.	Third Trial.
H	H	H
H	H	T
H	T	H
H	T	T
T	T	T
T	T	H
T	H	T
T	H	H

Here it is obvious that only one, out of the eight possible combinations which arise, favours the happening of the required event, viz., a head three times in succession. This, it will be seen, is the product of the three independent probabilities, viz.—

$$\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}.$$

Hence it is clear that the probabilities of independent events happening together, depend on the number of combinations into which

a							
b	*ab*						
c	*ac* *bc*	*abc*					
d	*ad* *bd* *cd*	*abd* *acd* *bcd*	*abcd*				
e	*ae* *be* *ce* *de*	*abe* *ace* *bce* *ade* *bde* *cde*	*abce* *abde* *acde* *bcde*	*abcde*			
f	*af* *bf* *cf* *df* *ef*	*abf* *acf* *bcf* *adf* *bdf* *cdf* *aef* *bef* *cef* *def*	*abcf* *abdf* *acdf* *bcdf* *abef* *acef* *bcef* *adef* *bdef* *cdef*	*abcdf* *abcef* *abdef* *acdef* *bcdef*	*abcdef*		
g	*ag* *bg* *cg* *dg* *eg* *fg*	*abg* *acg* *bcg* *adg* *bdg* *cdg* *aeg* *beg* *ceg* *deg* *afg* *bfg* *cfg* *dfg* *efg*	*abcg* *abdg* *acdg* *bcdg* *abeg* *aceg* *bceg* *adeg* *bdeg* *cdeg* *abfg* *acfg* *bcfg* *adfg* *bdfg* *cdfg* *aefg* *befg* *cefg* *defg*	*abcdg* *abceg* *abdeg* *acdeg* *bcdeg* *abcfg* *abdfg* *acdfg* *bcdfg* *abefg* *acefg* *bcefg* *adefg* *bdefg* *cdefg*	*abcdeg* *abcdfg* *abcefg* *abdefg* *acdefg* *bcdefg*	*abcdefg*

their total possibilities can be arranged; and thus we arrive at the Doctrine of Combinations, or at least a few elementary considerations connected with that doctrine.

The Doctrine of Combinations teaches the mode of ascertaining the number of groups, such as ones, or unities; twos, or couplets; threes, or triplets, &c. &c., which can be arranged out of, or formed with, n persons, things, or symbols. Adopting Bernoulli's well known mode of arranging groups of symbols—taking, for example, the first seven letters of the alphabet—we obtain the preceding table. Here, as each new letter is added to the previous groups, it is first written by itself, and then in successive combination with all the groups preceding it, it is sufficiently obvious that such a mode of arrangement secures every combination possible with seven letters; viz., ones, twos, threes, fours, fives, sixes, and sevens.

The first perpendicular column exhibits the number of single letters, or unities, which is manifestly always equal to the number of symbols employed. The second perpendicular column shows the total number of couplets. The third the number of triplets. The fourth the number of fours, and so on; and the seventh, or last, exhibits the number of sevens, which, in this case, is obviously equal to unity.

It will be convenient to adopt the following notation:—

Put $\int \overline{abc}^{\,1} \ldots 7$ equal to the sum of the first column;

$\int \overline{abc}^{\,2} \ldots 7$ „ „ second column;

$\int \overline{abc}^{\,3} \ldots 7$ „ „ third column;

$\int \overline{abc}^{\,4} \ldots 7$ „ „ fourth column;

and so on: and also put

$\int_7 \overline{abc}^{\,1} \,.\, 7$ equal to the sum of all the columns taken together.

Now, if we substitute for the literal groups in the foregoing table, numbers equal to the number of groups contained in each column, and in each division of a column, we shall obtain a numerical table in the following form, the mode of generating the numbers in which will be sufficiently obvious:—

1						
1	1					
1	2	1				
1	3	3	1			
1	4	6	4	1		
1	5	10	10	5	1	
1	6	15	20	15	6	1

A very brief inspection of this numerical table will show that the several perpendicular columns consist of several well-known series, each of which has a law by means of which it is generated, and may be continued to n terms. Those who are moderately skilled in the summation of series, will have no difficulty in proving that the sum of the first column is $= 7$;

$$\text{Of the 2nd column} = \frac{7 \cdot 6}{2} = 21;$$

$$\text{Of the 3rd column} = \frac{7 \cdot 6 \cdot 5}{2 \cdot 3} = 35;$$

$$\text{Of the 4th column} = \frac{7 \cdot 6 \cdot 5 \cdot 4}{2 \cdot 3 \cdot 4} = 35;$$

$$\text{Of the 5th column} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{2 \cdot 3 \cdot 4 \cdot 5} = 21;$$

$$\text{Of the 6th column} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} = 7;$$

$$\text{Of the 7th column} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} = 1.$$

And further, that the sum of all the columns taken together is $= 2^7 - 1 = 127$; and, as a further consequence, that as the law by

means of which each series is generated is the same for n symbols as for seven, it will follow that with n symbols there can be formed—

$$\text{Groups of ones} = n = \int \overline{abc}^{\,1} \ldots n;$$

$$\text{Groups of twos} = \frac{n \,.\, \overline{n-1}}{2} = \int \overline{abc}^{\,2} \ldots n;$$

$$\text{Groups of threes} = \frac{n \,.\, \overline{n-1} \,.\, \overline{n-2}}{2 \,.\, 3} = \int \overline{abc}^{\,3} \ldots n;$$

$$\text{Groups of fours} = \frac{n \,.\, \overline{n-1} \,.\, \overline{n-2} \,.\, \overline{n-3}}{2 \,.\, 3 \,.\, 4} = \int \overline{abc}^{\,4} \ldots n;$$

and so on: and the total number of combinations of all sorts, from unities to n' plets, will be equal to the sum of all the foregoing series, or equal to

$$n + \frac{n \,.\, \overline{n-1}}{2} + \frac{n \,.\, \overline{n-1} \,.\, \overline{n-2}}{2 \,.\, 3} + \frac{n \,.\, \overline{n-1} \,.\, \overline{n-2} \,.\, \overline{n-3}}{2 \,.\, 3 \,.\, 4} + \&c.,$$

$$= 2^n - 1 = \int_n \overline{abc}^{\,1} \ldots n.$$

Let us now examine the structure of the foregoing numerical table in reference to its *horizontal* columns. We shall ascertain that each of these columns represents the additional groups, which are generated by the addition of a new symbol, to any given number of preceding symbols; and each new column must, as a necessary consequence, contain the number of all the preceding groups taken together and one group added thereto, which additional group, or unity, is obviously the new symbol standing by itself. We shall further ascertain that each horizontal column represents the co-efficients of an expanded binomial of a power equal to one less than the number of terms, as thus:—

The first horizontal column is equal to the co-efficients of $\overline{a+b}^{\,0} = 1;$

The second is equal to $\overline{a+b}^{\,1} = 1+1;$

The third is equal to $\overline{a+b}^{\,2} = 1+2+1;$

The fourth is equal to $\overline{a+b}^{\,3} = 1+3+3+1;$

The fifth is equal to $\overline{a+b}^{\,4} = 1+4+6+4+1;$

and so on. Consequently the $n+1$'s horizontal column will be equal to

$$\overline{a+b}^{n} = 1+n+\frac{n\,.\,\overline{n-1}}{2}+\frac{n\,.\,\overline{n-1}\,.\,\overline{n-2}}{2\,.\,3}+\ \&c.$$

Which is manifestly equal to the total number of groups contained in the n preceding columns, plus unity.

(To be concluded in our next Number.)

On the Contrivances required to render Contingent Reversionary Interests Marketable Securities. By CHARLES JELLICOE, ESQ., *one of the Vice-Presidents of the Institute of Actuaries, &c.*

THIS subject has been so ably treated by Mr. Sang, in the paper transferred by Mr. Thomson's permission to the pages of this Magazine, that it might seem almost superfluous to revert to it. Nevertheless, there are considerations connected with it which I believe to be of some importance, and which that gentleman has not adverted to; and I am therefore induced to submit the following remarks, by way of pendant to his observations:—

To the majority of those I am addressing, no observation will probably appear more trite, than that whenever the idea of an average presents itself, we necessarily connect with it the notion of numbers,—the one qualification being an essential characteristic of the other; so that, in the case of unity, there can of course be no such thing as average. Obvious as is this distinction, it is somewhat remarkable that it has always been, and still is greatly overlooked, in dealing with the values of securities depending upon contingent or uncertain events. That is to say, the values in isolated cases have been confounded with those which are shown to exist when an average really obtains, although the conditions are as essentially different as can well be imagined. Thus the value of an annuity in an isolated case, on a single life, is even at the present time not unfrequently assumed to be identical with that which is found to be the average one amongst many such lives. The same may be said with regard to absolute and contingent reversions; and, as I have observed, with regard to almost all kinds of securities depending on contingent or uncertain events. The origin of so strange an oversight is probably to be found in the circumstance,